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Implication Algebras and the Metropolis–Rota Axioms for Cubic Lattices

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This paper is motivated by a result of Metropolis and Rota on an algebraic characterization of the lattice of faces of the n-cube (cubic lattice). Although their proof relies on an inductive argument, the axioms are independent of the dimension n. The question of how to extend this theory to include infinite cubic lattices was left open. We develop an extended characterization theory of cubic lattices of arbitrary dimension by adding three axioms (completeness, atomicity, and coatomicity) to those of Metropolis and Rota. The proof of our main theorem depends on the introduction of the *cubic implication algebra*, which is shown to satisfy Abbott's axioms for implication algebras. These algebras were first developed to characterize semi-Boolean algebras and Boolean algebras. © 1995 Academic Press, Inc.

1. INTRODUCTION

The algebraic characterization of Boolean lattices (both finite and infinite) have been extensively investigated (see, for example, [3–5]). A finite Boolean lattice has a geometric interpretation as the face lattice of

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an n-simplex in an n-dimensional Euclidean space. In high dimensions ($n \geq 5$) there are only three regular solids: the simplex, the cube, and the hyperoctahedron. Moreover, the face lattice of the n-cube and the face lattice of the n-hyperoctahedron are dual to each other and thus isomorphic; hence they yield only one new order structure which plays a complementary role to a Boolean lattice.

Metropolis and Rota [6] anticipated the existence of an analogous theory for the algebraic characterization of the face lattice of an n-cube. Indeed, they obtained a fundamental algebraic characterization of finite dimensional cubic lattices. Their theorem states as follows.

THEOREM 1.1 (Metropolis–Rota). *Let L be a finite lattice with minimum 0 and maximum 1. For every $x \neq 0$, let Δ_x be a function defined on the segment $[0, x]$ and taking values in $[0, x]$. Assume*

(i) *If $a \leq b \leq x$ then $\Delta_x(a) \leq \Delta_x(b)$;*

(ii) $\Delta_x^2 = \mathrm{id}$ (*the identity map*);

(iii) *Let $a < x$ and $b < x$. Then the following two conditions are equivalent:*

$$\Delta_x(a) \vee b < x \qquad \textit{and} \qquad a \wedge b = 0.$$

Then L is isomorphic to the lattice of faces of an n-cube, for some n.

Conversely, if L is the lattice of faces of an n-cube, and $\Delta_x(y)$ is antipodal face of y within the face x, then L satisfies conditions (i) *through* (iii).

The appealing nature of the above theorem lies in the fact that the axioms (i)–(iii) are independent of the dimension n of the n-cube, although proof of the above theorem in [6] is based on an induction on the dimension n. Naturally, Metropolis and Rota posed the question of characterizing infinite cubic lattices. The principal difficulty associated with this problem is finding a non-inductive proof technique.

First we observe that in addition to the Metropolis–Rota axioms three more axioms (completeness, atomicity, and coatomicity) are needed to characterize infinite cubic lattices. We show that for any lattice L (of arbitrary cardinality) satisfying these six axioms, there exists a set S such that L is isomorphic to the lattice $L(S)$ of signed sets of S.

The approach chosen here makes use of the implication algebra structure on the poset $L^+ = L \setminus \{0\}$, namely, the cubic implication algebra. Such algebras were originally used to characterize semi-Boolean algebras as well as Boolean algebras.

2. The Main Result

Several representations may be chosen for cubic lattices. In this paper a *cubic lattice is viewed as a lattice $L(S)$ of signed sets of a set S, where the* cardinality of S is arbitrary, defined as follows.

Definition 2.1 (Lattice of Signed-Sets, Infinite Cubic Lattice). Let S be a set of arbitrary cardinality. A signed set of S is a pair (A^-, A^-) of disjoint subsets of S.

$L^+(S)$ denotes the set of all such signed sets on S, ordered by the opposite of componentwise inclusion:

$$(A^+, A^-) \le (B^+, B^-) \qquad \text{if and only if } B^+ \subseteq A^+ \textit{ and } B^- \subseteq A^-.$$

$L(S)$ denotes the poset obtained from $L^+(S)$ by the adjunction of a minimum element 0.

It is easy to show that $L(S)$ forms a lattice with a maximum element $1 = (\emptyset, \emptyset)$ and a minimum element 0. Moreover, the join of $L(S)$ can be expressed as

$$(A^+, A^-) \vee (B^+, B^-) = (A^+ \cap B^+, A^- \cap B^-),$$

and when (A^+, A^-) and (B^+, B^-) are cross-disjoint signed sets, i.e., $A^+ \cap B^- = \emptyset$ and $A^- \cap B^+ = \emptyset$, then we have

$$(A^+, A^-) \wedge (B^+, B^-) = (A^+ \cup B^+, A^- \cup B^-),$$

otherwise we have

$$(A^+, A^-) \wedge (B^+, B^-) = 0.$$

Given a lattice L and an element $x \in L$, we shall use the notation $[x]$ to denote the principal filter generated by x, and (x) to denote the principal ideal generated by x. When x is a face of an n-cube, and y is a face inside x, then we use $\Delta_x(y)$ to denote the opposite face of y inside x. Suppose $x = (A^+, A^-)$, $y = (B^+, B^-)$, and $y \le x$. Then the diagonal operation Δ_x on the order ideal (x) is defined by

$$\Delta_x(y) = (A^+ \cup B^- \setminus A^-, A^- \cup B^+ \setminus A^+).$$

THEOREM 2.2. *If a lattice L, with 0 and 1, satisfies the following axioms:*

(A1) *For $x \in L$, there is an order-preserving map $\Delta_x: (x) \to (x)$;*

(A2) $\Delta_x^2 = \mathrm{id}$ *(the identity map);*

(A3) *If $0 < a, b < x$, then $a \vee \Delta_x(b) < x$ if and only if $a \wedge b = 0$;*

(A4) *L is complete;*

(A5) *L is atomistic: If $x \neq 0$ in L, then there is an atom $a \in L$ such that $a \leq x$;*

(A6) *L is coatomistic: Given $x \in L$, $x \neq 1$, there exist coatoms such that $x \leq s$,*

then L is isomorphic to the lattice $L(S)$ of signed sets of a set S with Δ_x serving as the diagonal map on (x).

Conversely, if L is the lattice $L(S)$ of signed sets of a set S, and $\Delta_x(y)$ is the antipodal face of y within the face x, then L satisfies conditions (A1) *through* (A6).

Throughout this paper we follow the terminology in [8]. Axioms (A1)–(A3) are exactly the Metropolis–Rota axioms; however, the lattices in [6] are all assumed to be finite, and as such automatically satisfy (A4)–(A6). We note that the axioms (A4)–(A6) are necessary for cubic lattices of arbitrary dimension. Let P be a lattice with 0 and 1. We may construct the P-signed set lattice $L(P)$ on the set of pairs (a^+, a^-) such that $a^+ \wedge a^- = 0$, where $a^+, a^- \in P$. The order relation of $L(P)$ and the diagonal operation Δ_x can be defined similarly. One may verify that $L(P)$ satisfies (A1)–(A3). However, if P is not complete, then neither is $L(P)$. Also, if P is not atomistic (coatomistic), then neither is $L(P)$.

Although it is not difficult to show that the lattice $L(S)$ of signed sets of S satisfies all the axioms (A1)–(A6), to prove the opposite direction of this statement is not just a verification; instead, we have to start from scratch and stick to the axioms. A major tool used in our proof is the theory of implications algebras. To make our proof self-contained, it appears necessary to give a brief review of implication algebras.

3. A BRIEF REVIEW OF IMPLICATION ALGEBRAS

The theory of implication algebras was originally developed by J. Abbott [1–3], in order to characterize semi-Boolean algebras as well as Boolean algebras by algebraic equations. In this section we shall briefly review the theory of implication algebras and include the proof of the theorem that is used to prove our main result.

DEFINITION 3.1 (Axioms for Implication Algebras). An implication algebra is a set A with a binary operation, called implicational product, $(a, b) \to ab$ satisfying the following axioms:

(I1) $(ab)a = a$,

(I2) $(ab)b = (ba)a$,

(I3) $a(bc) = b(ac)$.

LEMMA 3.2. *In any implication algebra, the following identities hold true*:

(i) $a(ab) = ab$,

(ii) $aa = bb$,

(iii) *There exists a unique element* 1 *in* A *such that for every* $a \in A$, $aa = 1$, $1a = a$, *and* $a1 = 1$.

Proof. (i) Using (I1) twice, we have $a(ab) = ((ab)a)(ab) = ab$.

(ii) Using (I1), (I2), and then (i), we get

$$aa = [(ab)a]a = [a(ab)](ab) = (ab)(ab). \tag{3.1}$$

Similarly, we have $bb = (ba)(ba)$. Thus, using (3.1) twice we get $aa = (ab)(ab) = [(ab)b][(ab)b]$. It follows from (I2) and (3.1) that $aa = [(ba)a][(ba)a] = (ba)(ba) = bb$.

(iii) Since aa is independent of a, we may denote it by 1. By (I1), $1a = (aa)a = a$, and by (i), $a1 = a(aa) = aa = 1$. ∎

PROPOSITION 3.3. *Let A be an implication algebra. Define the relation $\leq$ by $a \leq b$ if $ab = 1$. Then $(A, \leq)$ is a partially ordered set with greatest element* 1.

Proof. $a1 = 1$ immediately shows that $a \leq 1$ for every $a \in A$. The relation is reflexive since $aa = 1$ by definition. It is transitive as well: Suppose that $a \leq b$ and $b \leq c$, i.e., $ab = 1$ and $bc = 1$. Then

$$ac = a(1c) = a[(bc)c] = a[(cb)b] = (cb)(ab) = (cb)1 = 1.$$

Hence $a \leq c$. Lastly, suppose that $ab = ba = 1$. Then $a = 1a = (ba)a = (ab)b = 1b = b$. This shows that $\leq$ is anti-symmetric, and the proof is complete. ∎

From now on we shall keep $\leq$ for the partial order defined above for an implication algebra A. Moreover, by an implication algebra we also mean the partially ordered set based on the implication algebra.

LEMMA 3.4. *Let A be an implication algebra. Then we have*

(i) *$a \leq b$ if and only if $b = xa$ for some $x \in A$. In particular, the principal order theoretic filter $[a]$ consists precisely of all left multiples of a.*

(ii) *A is a join-semilattice and $a \vee b = (ab)b$.*

(iii) *If two elements a, b have a common lower bound p in A, then they have infimum given by $a \wedge b = [(ap) \vee (bp)]p$. In particular, any principal filter $[a]$ is a lattice.*

Proof. (i) If $a \leq b$, i.e., $ab = 1$, then $b = 1b = (ab)b = (ba)a$. Conversely, if $b = xa$, then $ab = a(xa) = x(aa) = x1 = 1$, so that $a \leq b$.

(ii) Since $a[(ab)b] = (ab)(ab) = 1$, we have $a \leq (ab)b$. Similarly, $b \leq (ba)a = (ab)b$, so that $(ab)b$ is an upper bound for a and b. Let $a, b \leq c$. Then we may write c as $(ca)a$. It follows that

$$[(ab)b]c = [(ab)b][(ca)a] = (ca)[[(ab)b]a] = (ca)[[(ba)a]a].$$

We now compute the factor $[(ba)a]a$:

$$[(ba)a]a = [a(ba)](ba) = [b(aa)](ba) = (b1)(ba) = 1(ba) = ba.$$

Therefore,

$$[(ab)b]c = (ca)(ba) = b[(ca)a] = bc = 1.$$

This implies that $(ab)b \leq c$, that is, $(ab)b$ is a least upper bound for a and b. Thus, A is a join-semilattice.

(iii) Note that the right implications product in A is antitone: $a \leq b$ implies $bc \leq ac$. Indeed, assume $ab = 1$ (or $a \leq b$). Then

$$(bc)(ac) = a[(bc)c] = a[(cb)b] = (cb)(ab) = (cb)1 = 1.$$

Suppose p is a common lower bound of a and b. Since $ap \leq ap \vee bp$, we see that $(ap \vee bp)p \leq (ap)p = a \vee p = a$. The same argument shows that $(ap \vee bp)p \leq b$; thus, $(ap \vee bp)p$ is a common lower bound of a and b. Let c be any common lower bound of a and b. We have $ap \leq cp$, $bp \leq cp$, and $ap \vee bp \leq cp$; therefore, $c \leq c \vee p = (cp)p \leq (ap \vee bp)p$, and the proof follows. ∎

PROPOSITION 3.5. *Let A be an implication algebra and $p \in A$. Then the principal filter $[p]$ is a complemented lattice under the map $a \to ap$ for $a \geq p$.*

Proof. First we have

$$ap \vee a = [(ap)a]a = aa = 1. \tag{3.2}$$

Next, note that $p \leq ap$ (by (i) of Lemma 3.4). By (iii) of Lemma 3.4, $a \wedge ap = [(ap)p]p$. Since $p \leq a$, we have $(ap)p = (pa)a = 1a = a$; thus $a \wedge ap = (ap \vee a)p = 1p = p$. ▮

LEMMA 3.6. *In any implication algebra A, principal filters $[p]$ are distributive.*

Proof. We subdivide the proof into four steps as follows:

(i) The map $a \to ca$ is isotone for every $c \in A$: Indeed, $a \leq b$ implies that $b = xa$ for some x; therefore, $cb = c(xb) = x(cb)$, leading to $ca \leq cb$.

(ii) If $a, b \in A$ are such that $a \wedge b$ exists, then $ab = a(a \wedge b)$: Let $p \leq a, b$. We claim that $ab = ap \vee b$. Since $p \leq b$, we have $ap \leq b$. On the other hand, $b \leq ab$, so that $ap \vee b \leq ab$. Let $c = ap \vee b$. Then $b \leq c$, so that $c = b \vee c = (cb)b$. By (3.2), it follows that $(ac)c = a \vee c = a \vee ap \vee b = 1$. Also, $[(ac)c]c = ac \vee c = ac$. Therefore,

$$\begin{aligned}(ab)c &= (ab)(1c) = (ab)([(ac)c]c) = (ab)(ac) = (ab)(a[(cb)b])\\ &= (ab)[(cb)(ab)] = (cb)[(ab)(ab)] = (cb)1 = 1.\end{aligned}$$

Thus, $ab \leq c$ implying $ab = c = ap \vee b$. Since $bp \geq p$, from (iii) of Lemma 3.4 it follows that $[a(bp)]p = (ap \vee bp)p = a \wedge b$. Hence

$$a(a \wedge b) = a([a(bp)]p) = [a(bp)](ap) = [b(ap)](ap) = b \vee ap = ab.$$

(iii) Let $p \in A$ and $a, b, c \in [p]$. Set $r = (a \wedge b) \vee (a \wedge c)$. Then we have $b \vee c \leq ar$: Since $r \geq a \wedge b$, we obtain $b(ar) \geq b[a(a \wedge b)] = b(ab) = 1$, showing $b \leq ar$. The same argument applies to c and the inequality is proved.

(iv) Finally, we have the distributivity: $r = (a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$. By (iii) and the antitone property of right implicational product, we have $(b \vee c)r \geq (ar)r = (ra)a = 1a = a$ because $r \leq a$. Therefore, $ar \vee (b \vee c)r \geq ar \vee a = [(ar)a]a = aa = 1$. Since $a \geq r$ and $(b \vee c) \geq r$, by (iii) of Lemma 3.4 we have

$$a \wedge (b \vee c) = [ar \vee (b \vee c)r]r = 1r = r = (a \wedge b) \vee (a \wedge c).$$

The proof is complete. ▮

We now come to the theorem of Abbott [3], which forms a crucial part in the proof of our main theorem.

THEOREM 3.7. *Let A be an implication algebra. Then for any $p \in A$, the principal filter $[p]$ is a Boolean lattice in the sense that it is complemented and distributive.*

4. Proof of the Main Theorem

As far as proof is concerned, this paper really starts from here. We have noted that any lattice $L(S)$ of signed sets based on a set S satisfies the axioms (A1)–(A6). So our task is to show that axioms (A1)–(A6) force L to be a lattice of signed sets. From now on, we shall assume that L is a lattice defined by the axioms (A1)–(A6), and Δ_x is also defined in the axioms.

Basically the Axioms (i) and (ii) in the Metropolis–Rota Theorem say that Δ_x is an order preserving involution of (x). Suppose $a < x$. Then $\Delta_x(a) < \Delta_x(x) = x$ and $\Delta_x \vee \Delta_x(a) < x$. Now by Axiom (iii), we have

$$a \wedge \Delta_x(a) = 0. \tag{4.1}$$

This relation was listed as a separate axiom in [6]. We also note the following immediate consequence of Axiom (A3): If $x \neq 0$ and $0 < a < x$, then

$$a \vee \Delta_x(a) = x. \tag{4.2}$$

The following notations are used in the rest of this paper: V is the set of atoms of L; C is the set of coatoms of L. For $x \in L$, V_x is the set of atoms less than or equal to x, and C_x is the set of coatoms greater than or equal to x. In addition, set $\Delta_1 = \Delta$.

LEMMA 4.1. *L is atomic.*

Proof. Let $x \neq 0$. By (A5), $V_x \neq \varnothing$ and by (A3), if $p \in V_x$, then $p \vee \Delta_x(p) = x$. Since $\Delta_x(p)$ is also an atom, the proof is complete. ∎

LEMMA 4.2. *The Δ_x operations are uniquely determined by axioms (A1)–(A3).*

Proof. Suppose Δ'_x satisfies (A1)–(A3). By (A1) and (A2), both Δ'_x and Δ_x are order-isomorphisms of (x). Because L is complete (A4) and the isomorphisms preserve joins (sups), they are hence determined by their values on the atoms of L and as such are less than or equal to x. So let $a \in V_x$. Then both Δ'_x and Δ_x are atoms, $\Delta'_x(a) \leq x$ and $\Delta_x(a) \leq x$. Now, either $\Delta'_x(a) = \Delta_x(a)$ or $\Delta'_x(a) \wedge \Delta_x(a) = 0$. Assume that the latter is true: By axiom (A3), we have $\Delta'_x(a) \vee \Delta_x(\Delta_x(a)) = a \vee \Delta'_x(a) < x$, which leads to the desired contradiction. Therefore, the second alternative is false, and it follows that $\Delta'_x = \Delta_x$. ∎

LEMMA 4.3. *For any atom a and any coatom s of L, either $a \leq s$ or $a \leq \Delta(s)$, but not both.*

Proof. Suppose $a \leq s$ and $a \leq \Delta(s)$, that is, $a \leq s$ and $\Delta(a) \leq s$. Then $1 = a \vee \Delta(a) \leq s$, which is an obvious contradiction. Next suppose $a \nleq s$. Since s is a coatom, we have $a \vee s = 1$. Now by (A3), $a \wedge \Delta(s) > 0$ and hence $a \leq \Delta(s)$ since by assumption a is an atom. ∎

The above lemma turns out to be sufficient to characterize coatoms of L:

LEMMA 4.4. *If $0 < s < 1$ in L has the property that for every atom a, either $a \leq s$ or $a \leq \Delta(s)$, then s is a coatom.*

Proof. Suppose $s < z$. Then there is an atom p such that $p \nleq s$ and $p < z$. Since $p \nleq s$, we have $\Delta(p) \leq s$ by Lemma 4.3, and so $p \vee \Delta(p) \leq z$. By (4.2) we also have $p \vee \Delta(p) = 1$, hence $z = 1$; and therefore s is a coatom. ∎

We remark that in the above proof of Lemma 4.4, the following fact is implicitly used: In any atomic lattice, suppose $y \nleq x$, then there exists an atom a such that $a \nleq x$ and $a \leq y$.

PROPOSITION 4.5. *L is coatomic, and for any $x > 0$ we have $x = \inf(C_x)$.*

Proof. Let $x > 0$. By (A6), $C_x \neq \varnothing$. Next by (A4), $\inf(C_x)$ exists. We wish to show that $x = \inf(C_x)$. Suppose not, i.e., suppose that $x < \inf(C_x)$. Since L is atomic by an earlier result, there exists an atom a such that $a \nleq x$ and $a \leq \inf(C_x)$. Evidently since a is an atom, $a \nleq x$ suggests that $a \wedge x = 0$. Also, (A3) implies that $x \vee \Delta(a) < 1$. Now by (A6), there exists a coatom s such that $x \vee \Delta(a) \leq s$. Then we have $\Delta(a) \leq s$. Since s is a coatom and $s \geq x$, it follows that $s \in C_x$ and $s \geq \inf(C_x)$. Because of $a \leq \inf(C_x)$, we get $a \leq s$. Finally we are led to a contradiction: $1 = a \vee \Delta(a) \leq s$. ∎

The following proposition demonstrates that Δ_x as defined by the axioms for L is determined by Δ. Throughout we shall adopt the convention that $C_1 = \varnothing$ and $\inf(\varnothing) = 1$.

PROPOSITION 4.6. *The automorphism Δ, as defined on L determines the family of maps Δ_x for $x > 0$ in the following manner: If $0 < y \leq x$, then*

$$\Delta_x(y) = \inf\{x \wedge \Delta(s) | s \in C_y \setminus C_x\} = x \wedge \Delta(\inf(C_y \setminus C_x)). \quad (4.3)$$

Proof. For $x = 1$, Eq. (4.3) reduces to $\Delta(y) = \Delta(\inf(C_y))$, which is Lemma 4.5. So we may assume that $x < 1$.

Let a be an atom with $a \leq y$, and let $s \in C_y$, $s \notin C_x$, i.e., $y \leq s$ but $x \not\leq s$. We claim that $\Delta_x(a) \leq \Delta(s)$: If not, we have $\Delta_x(a) \leq s$ since $\Delta_x(a)$ is an atom and s is a coatom. Because $a \leq y \leq s$, we have $a \wedge \Delta_x(a) \leq s$, that is, $x \leq s$, a contradiction. Thus, we have shown that $\Delta_x(a) \leq \Delta(s)$ for any atom $a \leq y$ and any coatom $s \in C_y \setminus C_x$. Since L is atomic, and Δ_x and Δ are lattice isomorphisms, we obtain $\Delta_x(y) \leq (s)$ for any $s \in C_y \setminus C_x$, that is, $\Delta_x(y) \leq \Delta(\inf(C_y \setminus C_x))$. It follows that

$$\Delta_x(y) \leq x \wedge \Delta\big(\inf(C_y \setminus C_x)\big).$$

To show that the above inequality holds with equality, let a be an atom such that $a \leq x \wedge \Delta(\inf(C_y \setminus C_x))$. We claim $p \leq \Delta_x(y)$, or equivalently, $\Delta_x(a) \leq y$. If not; namely, $\Delta_x(a) \wedge y = 0$ (since a is an atom), we shall derive a contradiction. By axiom (A3), $y \vee a < x$. Since L is coatomic, there exists a coatom s such that $s \geq y \vee a$ but $s \not\geq x$. It follows that $a \leq \Delta(s)$. However, from $y \vee p \leq s$ we have $a \leq s$. This leads to a contradiction because we cannot have that both $a \leq \Delta(s)$ and $a \leq s$ hold for an atom a and a coatom s. ∎

COROLLARY 4.7. $\Delta\Delta_x(y) = \inf\{\Delta(x) \wedge s \mid s \in C_y \setminus C_x\} = \Delta(x) \wedge \inf\{s \mid s \in C_y \setminus C_x\}$.

We now come to the key construction of this paper—cubic implication algebra. Recall that L^+ is the semilattice obtained from L by removing the minimum element 0. For $x, y \in L^+$, we defined the implication product by

$$xy = y \vee \Delta\Delta_{x \vee y}(y). \tag{4.4}$$

First we show that the above definition of implication product is consistent with the order structure of L^+:

THEOREM 4.8. *For $x, y \in L^+$, $xy = 1$ if and only if $x \leq y$.*

Proof. If $x \leq y$, then $x \vee y = y$ and hence

$$xy = y \vee \Delta\Delta_{x \vee y}(y) = y \vee \Delta(y) = 1.$$

To prove the converse, suppose $x \not\leq y$. We wish to show that $xy < 1$. Note that $x \not\leq y$ is equivalent to $y < x \vee y$. Let $z = x \vee y$. The claim amounts to showing that there exists a coatom s such that $s \geq y \vee \Delta\Delta_z(y)$. To see this, recall that $\Delta\Delta_z(y) = \Delta(z) \wedge \inf(C_y \setminus C_x)$, which is less than or equal to $\inf(C_y \setminus C_x)$. Since $x \not\leq y$, there exists a coatom $s \in C_y \setminus C_x$. Clearly, such an element s is greater than or equal to $\inf(C_y \setminus C_x)$, as well as greater than or equal to y. It follows that $y \vee \Delta\Delta_z(y) \leq s < 1$. ▮

PROPOSITION 4.9. *Let $x, y \in L^+$. Then $C_{xy} = C_y \setminus C_x$, hence $xy = \inf(C_y \setminus C_x)$.*

Proof. If $xy = 1$, then $x \leq y$, and $C_y \subseteq C_x$, implying $C_y \setminus C_x = \varnothing$, which is consistent with conventions $\inf(\varnothing) = 1$ and $C_1 = \varnothing$.

Now we suppose that $xy < 1$. Since

$$\Delta_{x \vee y}(y) = (x \vee y) \wedge \Delta(\inf(C_y \setminus C_{x \vee y})) = (x \vee y) \wedge \Delta(\inf(C_y \setminus C_x)),$$

it follows that

$$\Delta(\inf(C_y \setminus C_x)) \geq \Delta_{x \vee y}(y). \tag{4.5}$$

Let s be any coatom in $C_y \setminus C_x$. By (4.5) we obtain

$$\Delta(s) \geq \Delta(\inf(C_y \setminus C_x)) \geq \Delta_{x \vee y}(y),$$

i.e., $s \geq \Delta\Delta_{x \vee y}(y)$. Since $s \geq y$, we conclude $s \geq xy$; this establishes that $C_y \setminus C_x \subseteq C_{xy}$.

There remains to show that $C_{xy} \subseteq C_y \setminus C_x$. Let $s \in C_{xy}$, i.e., $s \geq y$ and $s \geq \Delta\Delta_{x \vee y}(y)$. If $s \geq x$, it would follow that $s \geq x \vee y$ and hence $s \geq \Delta_{x \vee y}(y)$ because $\Delta_{x \vee y}(y) \leq x \vee y$. Hence $1 = \Delta_{x \vee y}(y) \vee \Delta\Delta_{x \vee y}(y) \leq s$, a contradiction; therefore $s \notin C_x$. We conclude that $C_{xy} \subseteq C_y \setminus C_x$. ▮

THEOREM 4.10. *L^+ forms an implication algebra.*

Proof. One needs to verify that for all $x, y, z \in L^+$,

(B1) $(xy)x = x$.

(B2) $(xy)y = (yx)x$.

(B3) $x(yz) = y(xz)$.

By Proposition 4.9, (B1)–(B3) are immediate from the following identities by taking the infimums, where $'$ stands for the complementation with respect to the set C of coatoms of L:

(1) $C_{(xy)x} = C_x \setminus C_{xy} = C_x \cap C'_{xy} = C_x \cap (C_y \cap C'_x)' = C_x \cap (C'_y \cup C_x) = C_x$.

(2) $C_{(xy)y} = C_y \setminus C_{xy} = C_y \cap (C_y \cap C'_x)' = C_y \cap (C'_y \cup C_x) = C_y \cap C_x$, and $C_{(yx)x} = C_x \setminus C_{yx} = C_x \cap (C_x \cap C'_y)' = C_x \cap (C'_x \cup C_y) = C_x \cap C_y$.

(3) $C_{x(yz)} = C_{yz} \setminus C_x = (C_z \cap C'_y) \cap C'_x$, and $C_{y(xz)} = C_{xz} \setminus C_y = (C_z \cap C'_x) \cap C'_y$. ▮

The implication algebra structure of L^+ leads us to conclude that every principal filter of L^+ forms a Boolean lattice in the sense that it is both complemented and distributive. Note that the term *Boolean lattice* is often used to refer to the lattice of subsets of a set S ordered by inclusion. Evidently such lattices are referred to as *power set algebras* or the *power set lattices* of the set S. In the case of finite lattices either interpretation may be asserted. However, for infinite Boolean lattices, there exists a gap between these two notions: In the representation theory of infinite Boolean lattices it is known that a Boolean lattice is isomorphic to some power set lattice if and only if it is both atomistic and complete (see [3]). Fortunately, our case is based upon the principal filters of L^+ and here these two concepts coincide with each other.

THEOREM 4.11. *For any $x \in L^+$, the principal filter $[x]$ is isomorphic to the power set lattice of C_x.*

Proof. Suppose we have shown that $[x]$ is isomorphic to a power set lattice, then it is clear that $[x]$ is isomorphic to C_x. The completeness of $[x]$ is inherited from L. Since L is coatomistic, so is $[x]$. Therefore, the dual lattice of $[x]$ is isomorphic to a power set lattice, and so is $[x]$. ∎

We are now ready to finish the proof of the main theorem of this paper.

Proof of Theorem 2.2. Let a be an atom of L and set $S = C_a$. It follows from Lemma 4.3 that $C = S \cup \Delta(S)$, with the property that $S \cap \Delta(S) = \varnothing$. The following observation will be used immediately: Suppose that $x, y \in L^+$ satisfy the relation $x \wedge y = 0$. There exists a coatom s such that $s \geq x$ and $\Delta(s) \geq y$. Indeed, $x \wedge y = 0$ implies $x \vee \Delta(y) < 1$ by (A3); any coatom $s \geq x \vee \Delta(y)$ will serve this purpose.

Define the map φ: $L(S) \to L$ by

$$\varphi((A^+, A^-)) = \inf(A^+) \wedge \inf(\Delta(A^-)), \qquad (A^+, A^-) \in L(S), \quad (4.6)$$

and $\varphi(0) = 0$. Note that $\inf(A^+)$ and $\inf(A^-)$ are strictly greater than 0 for any signed set $(A^+, A^-) \in L(S)$ because $A^+, A^- \subseteq S = C_a$. It is clear that φ is order-preserving and that the atoms $(S, \varnothing)$ and $(\varnothing, S)$ of $L(S)$ map to a and $\Delta(a)$, respectively.

On the other hand, we define ψ: $L \to L(S)$ by

$$\psi(x) = (C_x \cap S, \Delta(C_x) \cap S), \qquad x \in L^+, \qquad (4.7)$$

and $\psi(0) = 0$. For $x > 0$, it is clear that $C_x \cap \Delta(C_x) = \varnothing$, i.e., that $(C_x \cap S, \Delta(C_x) \cap S)$ is an element of $L^+(S)$. Moreover, $x \leq y$ is equivalent to $C_y \subseteq C_x$ and hence ψ is order-preserving.

Next, $x = \inf(C_x \cap S) \wedge \inf(C_x \cap \Delta(S))$ immediately shows that $\varphi(\psi(x)) = 1$ for $x \in L^+$; since $\varphi(\psi(0)) = 0$ as well, we conclude that

$$\varphi \circ \psi = 1_L. \tag{4.8}$$

Let $x = \varphi((A^+, A^-))$. By definition, $x = \inf(A^+ \cup \Delta(A^-))$. The fact that the principal filter $[x]$ is a power set algebra means that $[x]$ is irredundantly coatomic, i.e., every element in $[x]$ is uniquely representable as a meet of coatoms in C_x. In particular, if D is a set of coatoms such that $x = \inf(D)$, then D must be C_x because $x = \inf(D)$ implies that $D \subseteq C_x$. We now obtain

$$C_x = A^+ \cup \Delta(A^-).$$

It follows that $C_x \cap S = A^+$ and $\Delta(C_x) \cap S = A^-$ since $A^+, A^- \subseteq S$, $\Delta(A^+) \cap S = \varnothing$, and $\Delta(A^-) \cap S = \varnothing$. But this means that $\psi(x) = (A^+, A^-)$ and we conclude that

$$\psi \circ \varphi = 1_{L(S)}. \tag{4.9}$$

Combining (4.8) and (4.9) we reach the conclusion that φ is an order isomorphism, hence $L \simeq L(S)$.

Finally, we show that the choice of S is unique up to isomorphism. If there exists T such that $L(T) \simeq L$, then $L(S) \simeq L(T)$. Let α be an isomorphism from $L(S)$ to $L(T)$. Note that an atom of $L(S)$ is a signed set (A, B) such that $A \cup B = S$. Let (A, B) be such an atom of $L(S)$; then the image of (A, B) under α is an atom (X, Y) of $L(T)$. Clearly, we have $T = X \cap Y$. Next, consider the principal filter $[A, B]$ generated by (A, B) in $L(S)$ and the principal filter $[X, Y]$ generated by (X, Y) in $L(T)$. Notice that the map α induces an isomorphism between $[A, B]$ and $[X, Y]$. It follows that there also exists an isomorphism between the set of coatoms of $[A, B]$ and the set of coatoms of $[X, Y]$. Now, a coatom of $[A, B]$ is either of the form $(\{a\}, \varnothing)$ or $(\varnothing, \{b\})$, where $a \in A$ and $b \in B$; also, a coatom of $[X, Y]$ is either of the form $(\{x\}, \varnothing)$ or $(\varnothing, \{y\})$. Clearly, there exists a natural bijection between S and the set of coatoms of $[A, B]$. The same argument holds for T and the set of coatoms of $[X, Y]$. Thus, $S \simeq T$.

The uniqueness of Δ_x (Lemma 4.2) implies that Δ_x is indeed the diagonal map on (x). ∎

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REFERENCES

1. J. C. ABBOTT, Implication algebras, *Bull. Math. Soc. Sic. Math. R.S. Roumanie* **11**, No. 59 (1967), 1.
2. J. C. ABBOTT, Semi-Boolean algebra, *Mat. Vesnik* **19** (1967), 177–198.
3. J. C. ABBOTT, "Sets, Lattices, and Boolean Algebras," Allyn & Bacon, Boston, MA, 1969.
4. G. BIRKHOFF, Lattice theory, *in* "American Mathematical Society Colloquium Publications," Vol. 25, Amer. Math. Soc., Providence, RI, 1979.
5. G. GRATZER, "Lattice Theory," Academic Press, New York, 1978.
6. N. METROPOLIS AND G.-C. ROTA, Combinatorial structure of the faces of the *n*-cube, *SIAM J. Appl. Math.* **35** (1978), 689–694.
7. J. S. OLIVEIRA, "The Theory of Cubic Lattices," Ph.D. thesis, MIT, 1992.
8. R. STANLEY, "Enumerative Combinatorics," Vol. I, Wadsworth & Brooks, Monterey, CA, 1986.