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Absolute and uniform convergence of alternate forms of the prolate spheroidal radial wave functions

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Abstract

A new orthonormal basis set representation of the prolate spheroidal radial and angular wave functions is presented. The embedded series solutions to a fully-coupled fluid-solid interaction continuum physics problem is defined by product sets of Legendre polynomials and modified spherical Bessel functions of the first and third kinds. We prove that the embedded series solutions analytically converge absolutely and uniformly to the exact solutions of the system of coupled continuum equations. The satisfaction of the bilinear concomitant and its utility in establishing the convergence proofs is demonstrated.

1. Background

A transient solution was presented by Jones-Oliveira [4,6] which models the fluid-solid interaction of a thin elastic prolate spheroidal shell loaded end-on by a nonconservative acoustic shock wave. Solutions to the Lagrangian equations of motion were provided for the normal and tangential shell displacement fields, as well as for the incident, scattered and radiated fluid pressure fields. The explicit analytic solutions were claimed to converge absolutely and uniformly to the exact

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solution of the actual coupled differential equations over the entire temporal and spatial domains both in the structure and in the fluid. However, proofs were promised which are now offered in completion of the work.

1.1. Problem description

The problem addressed was to solve the fluid-solid interaction problem for the prolate spheroidal geometry. A neutrally buoyant prolate spheroidal shell structure is assumed to be submerged in an acoustic medium that is initially at rest and whose far field is assumed to be well-defined and to remain at rest. The thin elastic shell is loaded end-on by an incident pressure wave Π^{inc}, which reduces the model complexity due to the axial symmetry of the physics of the problem. The incident wave impinges on the shell as if it were a rigid body. This incoming wave is modified by the shell, which acts as an obstacle in the path of the incident wave, producing an outward scattered or reflected wave Π^{sca}. Once the incident wave has struck the deformable shell, radiated or interactive vibratory waves Π^{rad} are initiated. The shell and the fluid behaviors interact and are inextricably linked by the kinematic boundary conditions at the fluid-solid interface.

To investigate completely this transient three-dimensional continuum mechanics problem, it was necessary to obtain simultaneous solutions for the incident pressure loading, the transient wave scattering and the dynamic and radiation problems associated with the shell and the fluid. The physical domain addressed relates to both the transient response of the shell structure and to the propagation of acoustic energy into/from the surrounding medium. This type of interactive transient problem is modeled as a mixed initial-boundary value problem which is described by coupled inhomogeneous partial differential equations.

The analysis paralleled that of its spherical analog, as presented in Jones-Oliveira et al. [5,7], which facilitated the development and verification of the solution by way of limiting arguments, cf. Jones-Oliveira [6, Section 2.4.1.2] to reduce the prolate spheroidal expressions to their analogous spherical expressions. The prolate spheroidal shell was modeled using the prolate spheroidal coordinate system such that the shell equations of motion are represented by the vector prolate spheroidal wave equation and the fluid is modeled via the scalar prolate spheroidal wave equation. The prolate spheroidal angular and radial wave functions were used to represent the shell displacement and fluid pressure fields. The prolate spheroidal functions were re-expressed in terms of equivalent representations that are infinite series expansions using Legendre polynomials and modified spherical Bessel functions of the first and third kinds. These novel representations, in addition to having other attributes, further facilitate simplification to the known spherical case.

1.2. Prolate spheroidal functions

The prolate spheroidal coordinate system (ϕ, η, ξ) is used where ϕ is the axis angle, η is the cosine of the polar angle, and ξ is the coordinate along the axis of rotation. An admissible transformation from the Cartesian coordinate system (x, y, z) to the right-handed prolate spheroidal coordinate system (ϕ, η, ξ) is given by the following: $x = f\sqrt{(\xi^2 - 1)(1 - \eta^2)}\cos\phi$, $y = f\sqrt{(\xi^2 - 1)(1 - \eta^2)}\sin\phi$, and $z = f\xi\eta$.

Bear in mind that both the prolate spheroidal angular and radial functions solve the same ordinary differential equations over different ranges of the variable, cf. Jones-Oliveira [6, formulas 2.5.2.12a and b], Meixner et al. [8, Chapter 3], and Flammer [1, formulas 2.2.6 and 2.2.7].

The prolate spheroidal angular functions of the first kind may be found in Flammer [1, formula 3.1.3a] or Jones-Oliveira [6, formula 2.6.5.4]. Due to the axial symmetry already mentioned and the invariance it implies, the order m reduces to zero and the expansions of the prolate spheroidal angular functions become expansions in terms of Legendre polynomials:

$$S_k(c, \eta) = \left(\sum_{n=0,1}^{\infty}\right)^{*} d_n^k(c) P_n(\eta) \tag{1}$$

in Flammer's notation, cf. Flammer [1]; where η is the cosine of the polar angle and $c = \frac{1}{2}kd$, where k is the wave number and d is the is the interfocal distance. Note that the asterisk associated with the summation sign indicates a parity rule, which states that the summation is over only even values of n when k is even, and over only odd values of n when k is odd. Note here that we use the following simplification of notation and set $d_n^k(c) = a_n$.

As per Flammer [1, formula 5.1.5], the prolate spheroidal radial functions (for the axisymmetric case) may be expressed in terms of the prolate spheroidal angular functions (for the axisymmetric case) as follows: Assume that

$$R_n(c, \xi) = \int_a^b K(\eta, \xi) S_n(c, \eta)\, \mathrm{d}\eta$$

where $K(\eta, \xi)$ is a kernel satisfying the condition

$$\left[\left(1 - \eta^2\right) S_n \frac{\partial K}{\partial \eta} - \left(1 - \eta^2\right) K \frac{\partial S_n}{\partial \eta}\right]_{\eta=a}^{\eta=b} = 0.$$

Let $K(\eta,\xi) = C\,\mathrm{e}^{\mathrm{i}c\eta\xi} = C\,\mathrm{e}^{-s\eta\xi}$ where C is an arbitrary constant and s is the Laplace variable. This kernel will guarantee convergence provided that $\Re(s\xi) > 0$. Then

$$R_n\big(-s^2,\xi\big) = \int_a^b C\,\mathrm{e}^{-s\xi\eta} S_n\big(-s^2,\eta\big)\,\mathrm{d}\eta$$

is a solution of the separated prolate spheroidal wave equation governing ξ, i.e. the radial differential equation, cf. Jones-Oliveira [6, formula 2.5.2.12b], provided that the bilinear concomitant given above vanishes at both limits a and b.

2. Convergence proofs

We prove that two linearly independent prolate spheroidal wave functions, which are solutions to the acoustic wave equation, may be expanded in terms of the modified spherical Bessel functions of the first and third kinds, respectively. It must be shown that the integration and summation may be interchanged within the integral transforms with kernels e^{-st}, where s is positive, over two different intervals of integration, where in one case the integral is also improper. A final technicality in the construction of these proofs is to show that the integral transform is differentiable as a function of s.

2.1. Construction of the requisite identity

Let (a_n) be a sequence of real numbers such that:

$$n\left|\frac{a_{n+1}}{a_n}\right| \to 0 \quad \text{as } n \to \infty. \tag{2}$$

Then, $\sum a_n z^n$ is an entire function of z, $z \in \mathbb{C}$. The necessity of the condition on a_n defined by (2) is demonstrated as follows.

In our problem set-up, the a_n are the coefficients in the expansions of the spheroidal wave functions given in terms of Legendre polynomials. According to Flammer [1], (1) admits the condition that

$$n\left|\frac{a_{n+1}}{a_n}\right| \sim \frac{c}{n^2}, \qquad c > 0 \text{ a constant.}$$

For this case, "$\sim$" defines asymptotic proportionality: There exists a constant $d > 0$, such that

$$\frac{|a_{n+1}/a_n|}{\frac{c}{n^2}} \to d \quad \text{as } n \to \infty.$$

Next, observe that

$$\frac{n|a_{n+1}/a_n|}{\frac{c}{n}} \to d,$$

which implies that

$$n\left|\frac{a_{n+1}}{a_n}\right| \to 0.$$

Remark 1. The sequence presented by (2) indicates that a_n tends to zero "faster than $\frac{1}{n!}$." We will revisit this detail below as an auxiliary argument. Note that (2) is not satisfied, e.g., by the exponential series; but, if $a_n = \frac{1}{(n!)^\alpha}$ with $\alpha > 1$, then (2) is satisfied. This fact demonstrates that there are non-trivial sequences that satisfy (2).

The main result, given assumption (2), is realized by the identity

$$\int_1^\infty \mathrm{e}^{-st}\Bigl(\sum a_n P_n(t)\Bigr)\,\mathrm{d}t = \sum a_n \int_a^\infty \mathrm{e}^{-st} P_n(t)\,\mathrm{d}t, \tag{3}$$

where P_n is the nth Legendre polynomial. Next, observe that for $t \geqslant 1$, $P_n(t) > 0$ since $P_n(1) = 1$ and the zeroes of P_n lie in $(-1, 1)$.

To obtain the main result, it is sufficient to majorize $\mathrm{e}^{-st}\sum|a_n|P_n(t)$ by an integrable positive function. To this end, we apply the identity

$$P_n(t) = \frac{1}{\pi}\int_0^\pi \bigl(t + \sqrt{t^2-1}\cos\theta\bigr)^n\,\mathrm{d}\theta, \tag{4}$$

that is commonly known as the first Laplace integral.

For $t \geqslant 1$, (4) yields a crude estimate for $P_n(t)$: $P_n(t) \leqslant (2t)^n$. We now use $\sum|a_n|(2t)^n$ to majorize the polynomial. Observe that this estimate shows that $\sum a_n P_n(t)$ is absolutely summable over $[1,\infty)$ and uniformly summable on intervals $[1,a]$. Therefore, the sum is a continuous function of t.

With these facts, we now consider the following observations: Set $b_n = 2^{2n}|a_n|$, so that $\sum|a_n|(2t)^n = \sum b_n(t/2)^n$. Since

$$\frac{b_{n+1}}{b_n} = 4\left|\frac{a_{n+1}}{a_n}\right|,$$

we have that

$$(n+1)\frac{b_{n+1}}{b_n} \to 0,$$

which follows from (2). In particular, there is a least natural number n_1, uniquely determined by the sequence (a_n), such that

$$(n+1)\frac{b_{n+1}}{b_n} < 1, \quad \text{for all } n \geqslant n_1,$$

i.e.,

$$b_{n+1} < \frac{b_n}{n+1}, \quad n \geqslant n_1.$$

Next, set $b = n_1! b_{n_1}$, such that $b_{n_1} = \frac{b}{n_1!}$. An easy argument then implies that

$$b_n < \frac{b}{n!} \quad \text{for } n \geqslant n_1. \tag{5}$$

Therefore,

$$\sum_{n_1}^{\infty} |a_n|(2t)^n = \sum_{n_1}^{\infty} b_n \left(\frac{t}{2}\right)^n < b\,\mathrm{e}^{\frac{t}{2}}.$$

On the other hand, the sum of the first n_1 terms is a polynomial in t and, consequently,

$$\mathrm{e}^{-\frac{t}{2}} \sum_{0}^{n_1-1} (\cdots) \to 0 \quad \text{as } t \to \infty.$$

In particular, this function is bounded by some M on $[1, \infty)$:

$$\mathrm{e}^{-\frac{t}{2}} \sum_{0}^{n_1-1} (\cdots) < M.$$

Combining the two inequalities yields

$$\mathrm{e}^{-t} \sum |a_n|(2t)^n \leqslant (M + b)\,\mathrm{e}^{-\frac{t}{2}}. \tag{6}$$

This now, in turn, shows that the left-hand side of the inequality is integrable over $[1, \infty)$. The proof of (3) follows from the above argument.

Remark 2. Extensions to complex variables and differentiability of complex functions are given by the following two conditions:

(a) Let $0 < \epsilon < \frac{1}{2}$. Then

$$\mathrm{e}^{-(1-\epsilon)t} \sum (\cdots) < (M + b)\,\mathrm{e}^{-(\frac{1}{2}-\epsilon)t}.$$

This shows that the left-hand side of (3) converges absolutely on $[1, \infty)$ for $s > \frac{1}{2}$. Therefore, the identity holds uniformly in all such real s, and thus also for complex s with $\Re(s) > \frac{1}{2}$.

(b) Now, for any $k \geqslant 1$, $t^k\,\mathrm{e}^{-t/2} \to 0$ as $t \to \infty$, and we observe that for such a k, the integral

$$\int_a^{\infty} t^k\,\mathrm{e}^{-st} \Big(\sum a_n P_n(t)\Big)\,\mathrm{d}t$$

converges in the same manner as was just established for the case where $k = 0$. Up to a factor $(-1)^k$ for integral k, these integrals are the successive derivatives with respect to s of the left-hand side of (3). From this fact, we establish that this left-hand side of the strict inequality is indefinitely differentiable in s, $s \in (\frac{1}{2}, \infty)$. It is, in fact, holomorphic in the half-plane $\Re(s) > \frac{1}{2}$, cf. the literature on Laplace transforms such as Gradshteyn and Ryzhik [2] and also below.

2.2. Convergence of the requisite identity and reversibility of summation and integration

We next provide a direct proof of the required convergence of the right-hand side of (3), by using the bounded estimate of $P_n(t)$:

$$\sum |a_n| \int_1^\infty \mathrm{e}^{-st} (2t)^n \, \mathrm{d}t$$

is uniformly summable for every $s \geqslant 1$. Factoring out the powers of 2, we obtain the identity

$$\sum c_n \int_1^\infty \mathrm{e}^{-st} (t)^n \, \mathrm{d}t, \quad c_n = 2^n |a_n|.$$

It follows that

$$(n+1)\frac{c_{n+1}}{c_n} \to 0 \quad \text{as } n \to \infty.$$

By setting $u = t - 1$, i.e., $t = u + 1$, we now obtain

$$\int_1^\infty \mathrm{e}^{-st} (t)^n \, \mathrm{d}t = \mathrm{e}^{-s} \int_0^\infty \mathrm{e}^{-su} (u+1)^n \, \mathrm{d}u.$$

The integral on the right-hand side is a Laplace transform and, as such, it is trivial to compute:

$$\int_0^\infty \mathrm{e}^{-su} (u+1)^n \, \mathrm{d}u = \sum \binom{n}{k} \frac{k!}{s^{k+1}}.$$

For $s \geqslant 1$,

$$\sum \binom{n}{k} \frac{k!}{s^{k+1}} \leqslant \sum \binom{n}{k} k! = n! \sum_0^n (m!)^{-1} < n!\,\mathrm{e}.$$

The constant e does not effect the series convergence. We now claim that $\sum c_n n! < \infty$. This follows immediately from the ratio test, and so it also follows that

$$(n+1)\frac{c_{n+1}}{c_n} \to 0.$$

The conclusion, which follows from above, is that the formal power series

$$\sum a_n \int_1^\infty \mathrm{e}^{-st} P_n(t)\,\mathrm{d}t$$

is both absolutely and uniformly summable for $s \geqslant 1$.

We next use these observations to obtain a simple expansion of the right-hand side

$$F(s) = \sum a_n \int_1^\infty \mathrm{e}^{-st} P_n(t)\,\mathrm{d}t$$

of (3) in powers of $\frac{1}{s}$. The absolute and uniform convergence of (3), for $s \geqslant 1$, now follows.

It is known from tables of integrals and Laplace transforms that

$$\int_1^\infty \mathrm{e}^{-st} P_n(t)\,\mathrm{d}t = \frac{\mathrm{e}^{-s}}{s} g_n(s),$$

where $g_n(s)$ is the "Bessel polynomial" given by

$$g_n(s) = \sum_k \frac{(n+k)!}{k!(n-k)!} \frac{1}{(2s)^k}, \tag{7}$$

which equals $K_{n+1/2}(s)$ up to a factor independent of n. We next introduce the variable $z = \frac{1}{s}$, and set

$$h_n(z) = \sum_k \frac{(n+k)!}{k!(n-k)!} \left(\frac{z}{2}\right)^k. \tag{8}$$

The above result implies that $\sum a_n h_n(z)$ is absolutely and uniformly summable in $|z| \leqslant 1$.

Set $f(z) = \sum a_n h_n(z)$. This function is holomorphic in the neighborhood of the unit disc $|z| \leqslant 1$ by classical theorems established in the theory of complex variables.

Remark 3. We are simply dealing with an instance of "normal convergence," also known as "continuous convergence" or "local uniform convergence." In the present context, this is just the "co-topology" or the topology of uniform convergence on compact sets.

Set

$$f(z) = \sum A_m z^m,$$

where A_m is the mth Taylor coefficient of f at $z=0$: $A_m = (\frac{1}{m!}) f^{(m)}(0)$. The radius of convergence of the series is $R > 1$ since $f(z)$ is holomorphic on the boundary $|z| = 1$ of the unit disc. The successive derivatives $f^{(m)}(0)$ are given by the Cauchy formulas; thus,

$$A_m = \frac{1}{2\pi \mathrm{i}} \int\limits_{|z|=1} \frac{\sum a_n h_n(z)}{z^{m+1}} \,\mathrm{d}z. \tag{9}$$

Since the convergence of the series is uniform on the unit circle, summation and integration may be interchanged to obtain:

$$A_m = \frac{1}{2\pi \mathrm{i}} \sum a_n \int\limits_{|z|=1} \frac{h_n(z)}{z^{m+1}} \,\mathrm{d}z. \tag{10}$$

There now remains the computation of the integrals and, by the construction of h_n, this is reduced to simple integrals of the form

$$\int\limits_{|z|=1} \frac{z^k}{z^{m+1}} \,\mathrm{d}z = \int\limits_{|z|=1} \frac{\mathrm{d}z}{z^{m-k+1}}.$$

Observe that such an integral is 0 unless $m-k+1=1$, i.e. $k=m$, in which case it equals $2\pi\mathrm{i}$. This fact yields the following formulas:

$$\int\limits_{|z|=1} \frac{h_n(z)}{z^{m+1}} \,\mathrm{d}z = 2\pi \,\mathrm{i} \frac{1}{2^m} \frac{(n+m)!}{m!(n-m)!}$$

for $m \leqslant n$, and the integral is 0 for $n < m$. Substitution this identity into (10) yields the following formula for the coefficients A_m:

$$A_m = \frac{1}{2^m} \sum_{n \geqslant m} a_n \frac{(n+m)!}{m!(n-m)!}. \tag{11}$$

Note that the absolute convergence of this series can be verified directly by the ratio test.

Combining all the preceding results demonstrates that

$$F(s) = \sum_{m=0}^{\infty} \left(\sum_{n \geqslant m} a_n \frac{(n+m)!}{m!(n-m)!} \frac{1}{(2s)^m} \right).$$

The convergence of the series expansion of the right-hand side is absolute and uniform for $s \geqslant 1$.

2.3. Bilinear concomitant

We now turn to the remaining issue of the "bilinear concomitant." We need to show that

$$\left[\left(t^2-1\right)\frac{\partial}{\partial t}\,\mathrm{e}^{-st} f(t)-\left(t^2-1\right)\mathrm{e}^{-st} f'(t)\right]_1^\infty=0, \tag{12}$$

where $[\cdots]_1^\infty$ means, of course,

$$\lim_{b\to\infty}[\cdots]_1^b,$$

and the notations used here explicitly correspond to those used in the preceding subsections.

For $t=1$, both terms of (12) are zero because of the factor t^2-1; therefore, there only remains the behavior of t as $t\to\infty$: We have to argue that the limit at ∞ also is zero, e.g. that both of the terms inside $[\cdots]$ tend to zero as $t\to\infty$. This is shown by the following:

By the crude estimates made in Section 2.1, let $\mathrm{e}^{-st} f(t)\to 0$ for $s>\frac{1}{2}$. The argument given there easily adjusts to also show that, e.g., $\mathrm{e}^{-st}t^2 f(t)\to 0$ as $t\to\infty$. This now leaves the behavior of $(t^2-1)f'(t)$ for large values of t:

Recall that for $t\geqslant 1$, $f(t)=\sum a_n P_n(t)$, where the coefficients a_n satisfy the condition

$$n\left|\frac{a_{n+1}}{a_n}\right|\to 0 \quad \text{as } n\to\infty,$$

as stated by (2) of Section 2.1. We first show that for $t>1$, $\sum a_n P'(t)$ converges absolutely, and that the convergence is uniform on $[1+\epsilon,\infty)$ for every $\epsilon>0$. This demonstrates that for $t>1$, $f'(t)=\sum a_n P'(t)$.

Now, $P'(t)=(\sqrt{t^2-1})^{-1}P_n^1(t)$ and it suffices to establish that $\sum a_n P_n^1(t)$ is absolutely summable. But,

$$P_n^1(t)=\frac{n+1}{\pi}\int_0^\pi\left(t+\sqrt{t^2-1}\cos\theta\right)^n\cos\theta\,\mathrm{d}\theta,$$

which is known as the second Laplace integral, cf. Whittaker and Watson [9, Section 15.61]. Next, use the same crude estimate used earlier to obtain $|P_n^1(t)|\leqslant(n+1)(2t)^n$, for $t\geqslant 1$. This fact implies the summability of $\sum|a_n||P_n^1(t)|$ for $t\geqslant 1$. Moreover, if $t\geqslant 1+\epsilon$, then $(\sqrt{t^2-1})^{-1}<\frac{1}{\epsilon}$ and it follows that the above series converges uniformly in any interval $[1+\epsilon,a]$. As indicated, this proves that $f'(t)=\sum a_n P'(t)$.

Next,

$$\left(t^2-1\right)f'(t)=\sum a_n\sqrt{t^2-1}\,P_n^1(t);$$

since for $t\geqslant 1$, $\sqrt{t^2-1}<t$. This fact leads to the "cheap" estimate:

$$\begin{aligned}(t^2-1)\left|f'(t)\right| &\leqslant \sum |a_n|(n+1)2^n t^{n+1}\\ &\leqslant \sum |a_n|(n+1)(2t)^{n+1}, \quad \text{for } t>1.\end{aligned}$$

Next, set $b_n = |a_n|(n+1)2^{2n+2}$. For every fixed $k = 0, 1, 2, \ldots,$

$$(n+k)\frac{b_{n+1}}{b_n} \to 0.$$

Since the left-hand side equals

$$(n+k)\frac{n+2}{n+1}\left|\frac{a_{n+1}}{a_n}\right|2^2,$$

this tends to zero by (2). So, it follows that

$$(n+2)\frac{b_{n+1}}{b_n} \to 0.$$

Since the above sequence is positive, there exists a least n_1 such that

$$b_{n+1} < \frac{b_n}{n+2}, \quad \text{for } n \geqslant n_1.$$

Set

$$b = \frac{b_{n_1}}{(n_1+1)!}.$$

An easy induction argument on n shows that

$$b_{n+1} < \frac{b}{(n+1)!}, \quad \text{for } n \geqslant n_1.$$

Next, observe that

$$\sum |a_n|(n+1)(2t)^{n+1} = \sum b_n\left(\frac{t}{2}\right)^{n+1}.$$

We conclude again that

$$\sum_{n_1} |a_n|(n+1)(2t)^{n+1} \leqslant b\,\mathrm{e}^{\frac{t}{2}}.$$

Furthermore,

$$\mathrm{e}^{-\frac{t}{2}}\sum_0^{n_1} |a_n|(n+1)(2t)^{n+1}$$

is bounded by some constant $M > 0$, and hence

$$(t^2-1)\left|f'(t)\right| \leqslant (M+b)\,\mathrm{e}^{\frac{t}{2}}.$$

Consequently, $\mathrm{e}^{-st}(t^2-1)|f'(t)| \to 0$ as $t \to \infty$, for every $s > \frac{1}{2}$ (and even uniformly for $s \geqslant \frac{1}{2}+\epsilon$). Accordingly, the second term in (12) also vanishes.

We need one further result to complete the proof. One may recall, e.g. from Flammer [1, p. 45], that in order to use the “bilinear concomitant” condition, we need to interchange integration and a certain differential operator over the interval in question. Since our interval is $[1, \infty)$, the usual continuity conditions on partial derivatives under the integral are insufficient to guarantee the admissibility of interchanging the order of the integral and differential operators. But, the required condition is obtained by the following argument:

Let

$$L_s = a(s)\frac{\mathrm{d}^2}{\mathrm{d}s^2} + b(s)\frac{\mathrm{d}}{\mathrm{d}s} + c(s)$$

be a second-order linear differential operator with continuous coefficients on $[1, \infty)$. Also, let $f(t)$ be the function considered so far. Then,

$$L_s \int_1^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t = \int_1^\infty L_s\,\mathrm{e}^{-st} f(t)\,\mathrm{d}t,$$

where under the integral, $\frac{\mathrm{d}}{\mathrm{d}s}, \ldots$, are to be replaced by partial differentiation operators $\frac{\partial}{\partial s}, \ldots$. To establish this, it suffices to obtain the identity with $\frac{\mathrm{d}}{\mathrm{d}s}$ and $\frac{\mathrm{d}^2}{\mathrm{d}s^2}$ in place of L_s.

Next, observe that

$$\frac{\partial}{\partial s}\,\mathrm{e}^{-st} f(t) = -\mathrm{e}^{-st} t f(t)$$

and hence,

$$\left|\frac{\partial}{\partial s}\,\mathrm{e}^{-st} f(t)\right| \leqslant \mathrm{e}^{-t} t\,\big|f(t)\big|, \quad \text{for } t \geqslant 1.$$

It follows from the results obtained in Section 2.1 that

$$\int_1^\infty \mathrm{e}^{-t} t\,\big|f(t)\big|\,\mathrm{d}t$$

converges and, therefore, by the Weierstrass M-test, that also

$$\int_1^\infty \frac{\partial}{\partial s}\mathrm{e}^{-st} f(t)\,\mathrm{d}t$$

converges uniformly for $s \geqslant 1$.

A standard theorem of analysis, cf. Hobson [3, Vol. II, Chapter 5, Art. 246], now implies that

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_1^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t = \int_1^\infty \frac{\partial}{\partial s}\mathrm{e}^{-st} f(t)\,\mathrm{d}t.$$

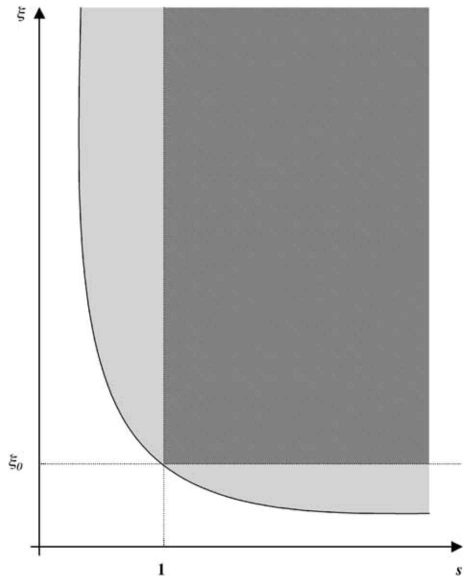

Fig. 1. Validity region.

The last integral equals

$$-\int_{1}^{\infty} \mathrm{e}^{-st} t f(t)\,\mathrm{d}t.$$

We now repeat the argument just used, since $\mathrm{e}^{-t}t^2|f(t)|$ also is integrable over $[1,\infty)$; thus, second derivatives with respect to s are also properly behaved and the result follows for L_s.

For $s \in \mathbb{R}$, the validity region for this argument is given by the inequality $s\xi \geqslant \xi_0$, depicted in Fig. 1. In particular, the region contains the rectangular region $\xi \geqslant \xi_0$, $s \geqslant 1$, with the boundary given by the dashed lines in Fig. 1.

For $s \in \mathbb{C}$, the validity region can now be visualized as follows: In $\mathbb{R}^3$, let the xy-plane be a copy of $\mathbb{C}$, i.e., $x = \Re(x)$ and $y = \Im(s)$. Then our region is bounded by the hyperbolic cylinder $\xi\Re(s) = \xi_0$ and consists of this surface together with its "outside" in the x-direction: This is the set of all points (x, y, ξ) with $x\xi \geqslant \xi_0$.

2.4. *Bilinear concomitant: connection to the spheroidal functions*

We conclude with some observations concerning the details of Flammer's [1] discussion of the prolate spheroidal radial wave function denoted by $S_n^{(3)}(-s^2, \xi)$ in Jones-Oliveira [6]. We claim that our remarks indicate why the "bilinear concomitant condition" is critical to our proofs of absolute and uniform series convergence as developed above. To begin, let

$$L_t = \frac{\mathrm{d}}{\mathrm{d}t}\left(1 - t^2\right)\frac{\mathrm{d}}{\mathrm{d}t} - k^2 t^2$$

be the differential operator, writing partial derivatives, of course, when applied to a function of several variables. Next, let L_s be the operator obtained by replacing t by s, and assume that the function $f(t)$ used thus far satisfies a differential equation of the form

$$L_t f + \lambda f = 0,$$

i.e., that it is an eigenfunction of L_t. Note that the eigenvalue λ will not matter in the sequel, except indirectly since it enters the computation of the expansion coefficients a_n in the earlier formulas. Write L_t^+ for $\frac{\mathrm{d}}{\mathrm{d}t}(1 - t^2)\frac{\mathrm{d}}{\mathrm{d}t}$, and similarly for L_s^+; and note that L_t^+ is the traditional Legendre differential operator. The formulas 2.5.2.2, etc. of Jones-Oliveira [6] show that in our coordinate system, the Laplacian Δ is written in the form

$$\Delta = \frac{1}{s^2 - t^2}\left(L_t^+ - L_s^+\right). \tag{13}$$

For our purposes, next consider a kernel of the form $K(s,t) = \mathrm{e}^{-ast}$. A simple calculation yields

$$\Delta K = a^2 K.$$

We also require the following condition:

$$\left(L_t^+ - L_s^+\right)K = 0. \tag{14}$$

It is easy to verify that (14) is equivalent to

$$\left(a^2 + k^2\right)K = 0. \tag{15}$$

If we let a be real, then this forces $k = \mathrm{i}a$ or $k = -\mathrm{i}a$. For our applications, $a = s$, which is in agreement with $k^2 = -s^2$.

To simplify matters, we assume that $a = 1$, i.e. that $k^2 = -1$, so that $L_t = L_t^+ + t^2$.

We know from results obtained earlier that $K(s,t) = \mathrm{e}^{-st}$; in particular,

$$L_s \int_1^\infty K(s,t) f(t)\,\mathrm{d}t = \int_1^\infty L_s K(s,t) f(t)\,\mathrm{d}t.$$

The integral on the right-hand side may also be written as

$$\int_1^\infty L_t K(s,t) f(t)\,\mathrm{d}t.$$

Note that (14) is used to yield the first one of Flammer's [1, p. 45] identities. Flammer's second identity,

$$\int_1^\infty \big[(L_t K)f - K(L_t f)\big]\,\mathrm{d}t = 0, \tag{16}$$

may be defined as follows: Let $K_t, K_{tt}, \ldots$ be the partial derivatives. Given this, observe that

$$\begin{aligned}(L_t K)f &= \big(1-t^2\big)K_{tt}f - 2tK_t f + t^2 K f,\\ K(L_t f) &= \big(1-t^2\big)Kf'' - 2tKf' + t^2 K f.\end{aligned}$$

The last terms in the two expressions are equal, so they drop out of $(L_t K)f - K(L_t f)$ and the difference becomes

$$\big(1-t^2\big)(K_{tt}f - Kf'') - 2t(K_t f - Kf'). \tag{17}$$

In order to rewrite the indefinite integral of the first summand, we next use integration by parts:

$$\begin{aligned}\int \big(1-t^2\big)K_{tt}f\,\mathrm{d}t &= \big(1-t^2\big)K_t f - \int K_t\big[\big(1-t^2\big)f\big]'\,\mathrm{d}t\\ &= \big(1-t^2\big)K_t f + 2\int tK_t f\,\mathrm{d}t - \int K_t\big(1-t^2\big)f';\end{aligned}$$

similarly,

$$\begin{aligned}\int \big(1-t^2\big)Kf''\,\mathrm{d}t &= \big(1-t^2\big)Kf' - \int \big[\big(1-t^2\big)K\big]_t f'\,\mathrm{d}t\\ &= \big(1-t^2\big)Kf' + 2\int tKf'\,\mathrm{d}t - \int \big(1-t^2\big)K_t f'\,\mathrm{d}t.\end{aligned}$$

Taking the difference of these expressions eliminates the last integrals on the respective second lines and this now yields

$$\begin{aligned}&\int \big(1-t^2\big)(K_{tt}f - Kf'')\,\mathrm{d}t\\ &\quad= \big(1-t^2\big)(K_t f - Kf') + 2\int t(K_t f - Kf')\,\mathrm{d}t.\end{aligned} \tag{18}$$

The first term on the right-hand side is precisely the "bilinear concomitant." Based upon our earlier results in Section 2.1, we begin by taking the definite integral $\int_1^\infty$ in (17), and observe that the first term on the right-hand side of (18) vanishes when evaluated between 1 and ∞. The second integral in (18) is the opposite of the second integral of (17). It now follows that (16) is verified.

3. Conclusions

In the work of Jones-Oliveira [4,6], t is replaced by η and s is replaced by $s\Xi = \underline{s}\xi$ where $\underline{s}$ is treated as a parameter. The variables of interest are ξ and η. The coefficients a_n are replaced by the $d_p^n(-s^2)$, where the summation index now is p rather than n. Finally, the integral kernel now is $\mathrm{e}^{-\underline{s}\xi\eta}$, up to the constant factor of $\frac{2}{\pi}$ chosen in formula 3.3.2.8 of Jones-Oliveira [6]. The function $f(t)$ thus becomes $S_n^{(3)}(-\underline{s}^2;\xi)$ as defined in formula 3.3.2.8 of Jones-Oliveira [6].

The alternate form for the prolate spheroidal radial function of the third kind is

$$S_n^{(3)} = \left(\sum_{p=0,1}^{\infty}\right)^* d_p^n\left(-\underline{s}^2\right)\sqrt{\frac{\pi}{2\Xi s}}K_{p+1/2}(\Xi s). \tag{19}$$

Similarly, the alternate form for the prolate spheroidal radial function of the first kind is

$$S_n^{(1)} = \left(\sum_{p=0,1}^{\infty}\right)^* d_p^n\left(-\underline{s}^2\right)\sqrt{\frac{\pi}{2\Xi s}}I_{p+1/2}(\Xi s). \tag{20}$$

The prolate spheroidal radial functions have now be written in terms of modified spherical Bessel functions of the first and third kinds. These alternative representations are computationally better behaved than the traditional functions, which are written in terms of continued fraction expansions and the spherical Bessel, (Neumann) and Hankel functions. In a sequel, we will be apply a similar set of transcendental basis functions to the rotationally symmetric cases of "Cassinian ovaloids."

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